Quadratic Expression and Equations

Expanding Expressions

Learning Intention

5A	**1. I can apply the distributive law to expand and simplify.** e.g. Expand and simplify $2x(3x - 5) - 3(3x - 5)$.	☐
5A	**2. I can expand a binomial product.** e.g. Expand and simplify $(2x - 3)(x + 4)$.	☐
5A	**3. I can expand to form a difference of two squares.** e.g. Expand $(3x + 2)(3x - 2)$.	☐
5A	**4. I can expand a perfect square.** e.g. Expand $(x + 5)^2$.	☐

Revisiting Expanding

We expand by using the **distributive law** to multiply terms and then combine **like terms**

- What is the distributive law?
  - $a(b + c) = ab + ac$
- What are like terms?
  - Terms that have the same pronumeral part, e.g. 7x and x, but not xy and x

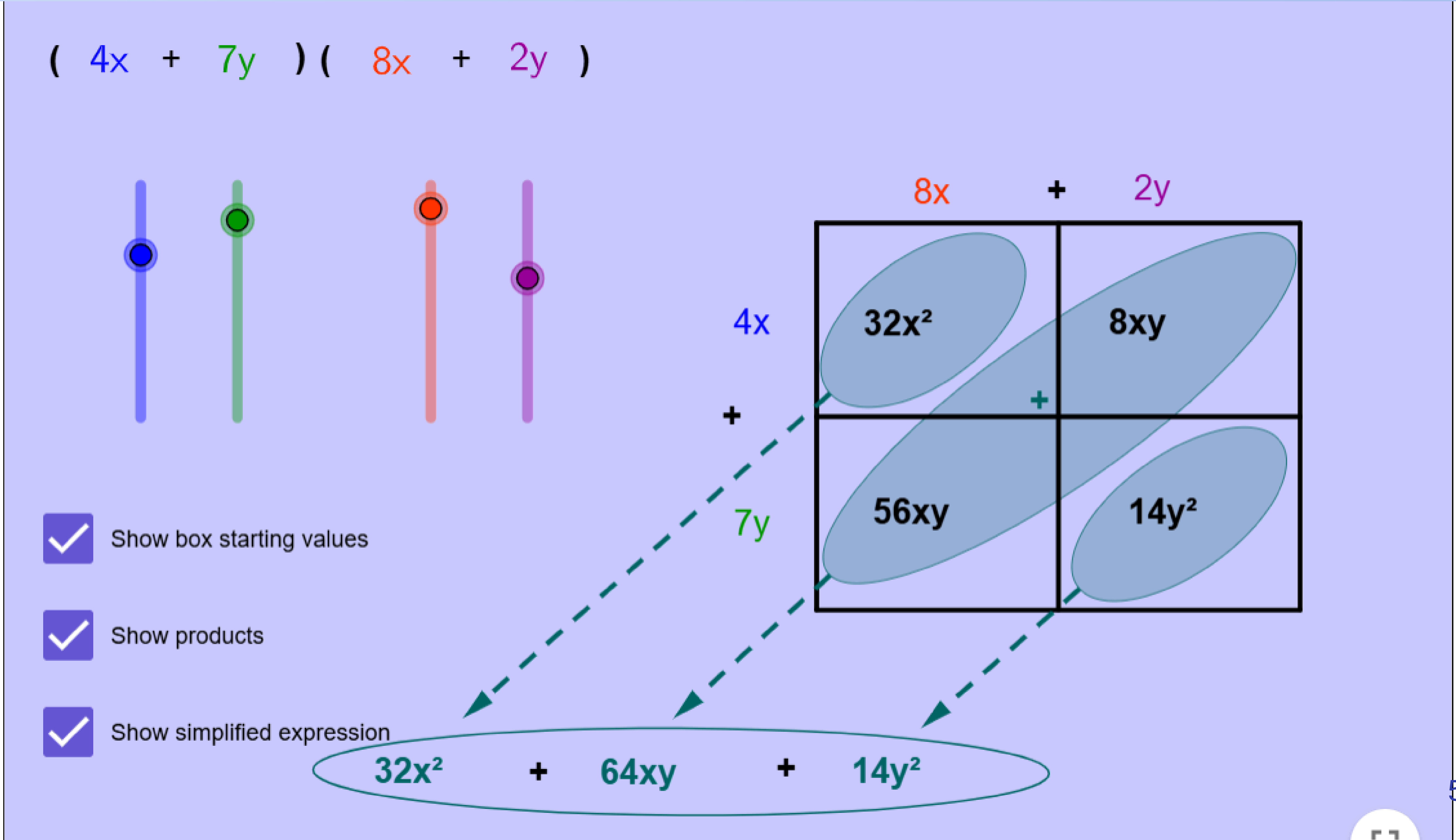
FOIL

We can multiply expressions like $(7x - 5)(-3x + 2)$ by following FOIL

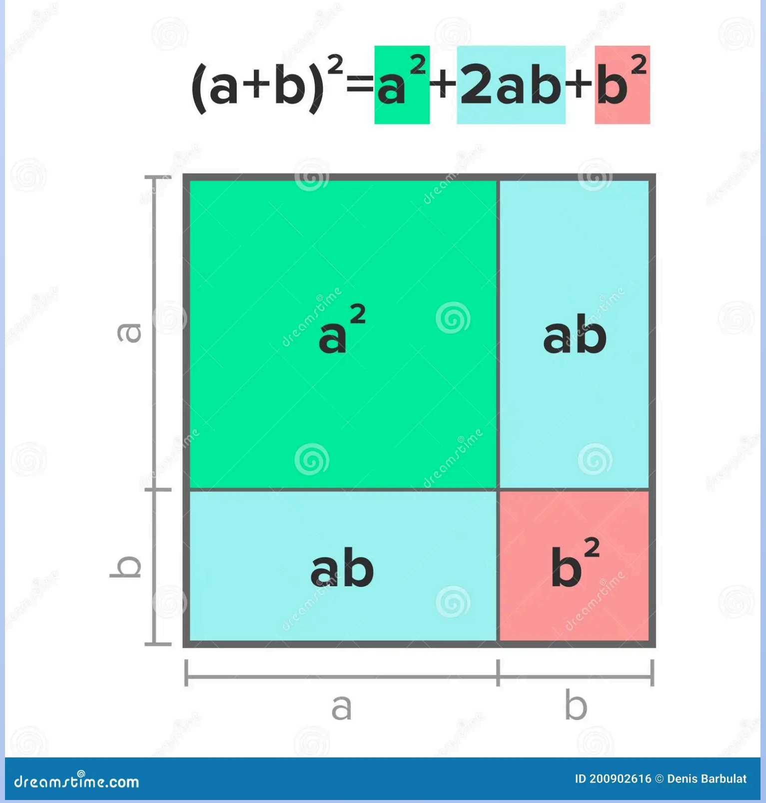
- **F**irst: $7x \times -3x$
  = $-21x^2$
- **O**utside: $7x \times 2$
  = $14x$
- **I**nside: $-5 \times -3x$
  = $15x$
- **L**ast: -5×2
  = -10

Then we add all the terms up and get:

- $-21x^2 + 29x - 10$


(4x + 7y) (8x + 2y)
8x + 2y
4x
+
7y
32x²
8xy
+
56xy
14y²
Show box starting values
Show products
Show simplified expression
32x² + 64xy + 14y²

Special case: Perfect Squares



$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

Because:

$(a+b)^2 = (a+b)(a+b)$

$= a^2 + ab + ab + b^2$

Example: $(3+5)^2$

$3^2 + 2 \times 3 \times 5 + 5^2$

$= 9 + 30 + 25$

$= 64$

$= 8^2$

What if we have -b instead?

$$(a-b)^2 = a^2 - 2ab + b^2$$

Because: $(a-b)^2 = (a-b)(a-b)$

$= a^2 - ab - ab + (-b)^2$

$= a^2 - 2ab + b^2$

Example $(7-5)^2$

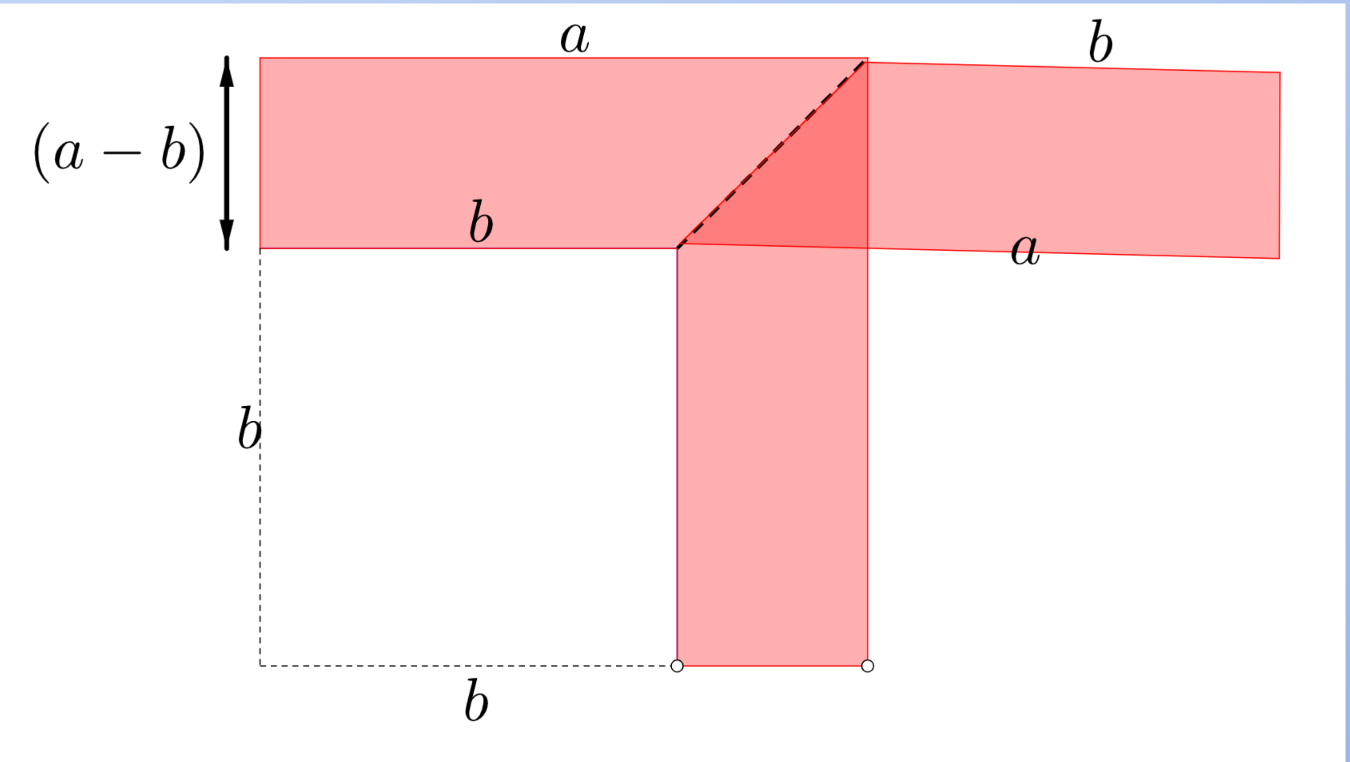
$7^2 - 2 \times 7 \times 5 + 5^2$

$= 49 - 70 + 25$

$= 4$

$= 2^2$

Special Case: Difference of squares



$$(a+b)(a-b) = a^2 - b^2$$

Because: $(a+b)(a-b) =$

$$a^2 - ab + ab - b^2$$

Mulitplying and Dividing Algebraic Fractions

5C	**9. I can cancel common factors in algebraic fractions.** e.g. Simplify $\frac{4x-2}{2}$.	☐
5C	**10. I can multiply and divide simple algebraic fractions.** e.g. Simplify $\frac{3x-9}{20} \div \frac{2x-6}{5}$.	☐

Find the mistake

a $\dfrac{6x - \cancel{8}^{2}}{\cancel{4}_{1}} = \dfrac{6x - 2}{1}$

$= 6x - 2$

b $\dfrac{2a}{9} \div \dfrac{2}{3} = \dfrac{2a}{9} \times \dfrac{2}{3}$

$= \dfrac{4a}{27}$

c $\dfrac{3b}{7} \div \dfrac{2b}{3} = \dfrac{3b}{7} \times \dfrac{3b}{2}$

$= \dfrac{9b^2}{14}$

Multiplying algebraic fractions

- We multiply the numerators with each other
- and multiply the denominators with each other
- Remember to add the indices of like pronumerals (e.g. $\frac{x^2}{y} \times \frac{x^3}{y^2} = \frac{x^5}{y^3}$)
- We cancel common factors between the numerator and denominator

Dividing Algebraic Fractions

- We multiply by the reciprocal of the fraction after the division sign
- What's the reciprocal of a?
  - $\frac{1}{a}$
- What's the reciprocal of $\frac{a}{b}$?
  - $\frac{b}{a}$
- Remember when we had to find the radius of a sphere?
  - $V = \frac{4}{3}\pi r^3$
  - $r^3 = V \times \frac{3}{4\pi}$

Factorising

5. I can factorise by taking out a common factor. e.g. Factorise $12x^2 - 18x$.	☐
6. I can factorise a difference of two squares. e.g. Factorise $9x^2 - 16$.	☐
7. I can factorise a difference of two squares involving surds. e.g. Factorise $x^2 - 7$ using surds.	☐
8. I can factorise using grouping. e.g. Factorise $x^2 - ax + 2x - 2a$ by grouping.	☐

Lesson starter: But there are no common factors!

An expression such as $xy + 4x + 3y + 12$ has no common factors across all four terms, but it can still be factorised, by what we call the method of grouping

- xy + 4x + 3y + 12 = x(______) + 3(______)
    - = (______)(x + 3)
- Now let's try it another way.
  xy + 3y + 4x + 12 = y(______) + 4(______)
    - = (______) (______)
- Are the two results the same?
- Let's summarize what we did: We arranged the four terms in pairs
- Then we saw that we could represent the two pairs as two sets of expanded expressions

There are 3 types of expressions we will factor:

1. Common factors

What's the common factor here?

- $-5x - 20$
  - = $-5(x + 4)$
- $4x^2 - 8x$
  - = $4x(x - 2)$

2. Difference of squares

We use $a^2 - b^2 = (a+b)(a-b)$ to factor these

- $x^2 - 16$
  - = $(x-4)(x+4)$

If one term is not a perfect square, we can use surds

- $5 - x^2$
  - = $(\sqrt{5} - x)(\sqrt{5} + x)$

3. Four-Term Expressions

Try to group terms so we can factorise pairs
(This may not always be possible)

- $xy + 5x - 2y - 10$
  - = $x(y+5) - 2(y+5)$
  - = $(x-2)(y+5)$

Factorising Monic Quadratic Trinomials

11. I can factorise a monic trinomial. e.g. Factorise $x^2 - 8x - 20$.	☐
12. I can factorise a trinomial with a common factor. e.g. Factorise $3x^2 - 24x + 45$.	☐
13. I can multiply and divide algebraic fractions by first factorising. e.g. Simplify by first factorising $\frac{x^2 - 4}{x + 2} \times \frac{3x + 12}{x^2 + 2x - 8}$.	☐

What do these words mean?

- **Monic**: The coefficient of the highest power of x is 1
- **Quadratic**: The highest power of x is 2
- **Trinomial**: There are 3 terms in the expression

What are my numbers?

- I've got two numbers a and b such that $a + b = 5$ and $ab = 6$
  - Answer: 2 and 3
- I've got two numbers a and b such that $a + b = -10$ and $ab = 21$
  - Answer: -3 and -7
- I've got two numbers a and b such that $a + b = 1$ and $ab = -20$
  - Answer: 5 and -4
- I've got two numbers a and b such that $a + b = -4$ and $ab = -21$
  - Answer: -7 and 3

How does this help us factorise?

Let's look at what happens when we expand $(x+a)(x+b)$

- By FOIL, we have:
- $(x+a)(x+b) = x^2 + bx + ax + ab$
- = $x^2 + (a+b)x + ab$
- So if we can find a and b
  then we can factorise the expression $x^2 + (a+b)x + ab$

Learning Intention

14. I can factorise a non-monic quadratic. e.g. Factorise $5x^2 + 13x - 6$.	☐

What if x^2 has a coefficient other than 1?

- Like $15x^2 - x - 6$
- That's what we call a "nonmonic quadratic trinomial"
- We multiply x^2's coefficient with the constant term
- And then play the $(a + b)$ and ab game
- And play the game again with a and b with the coefficient of x^2

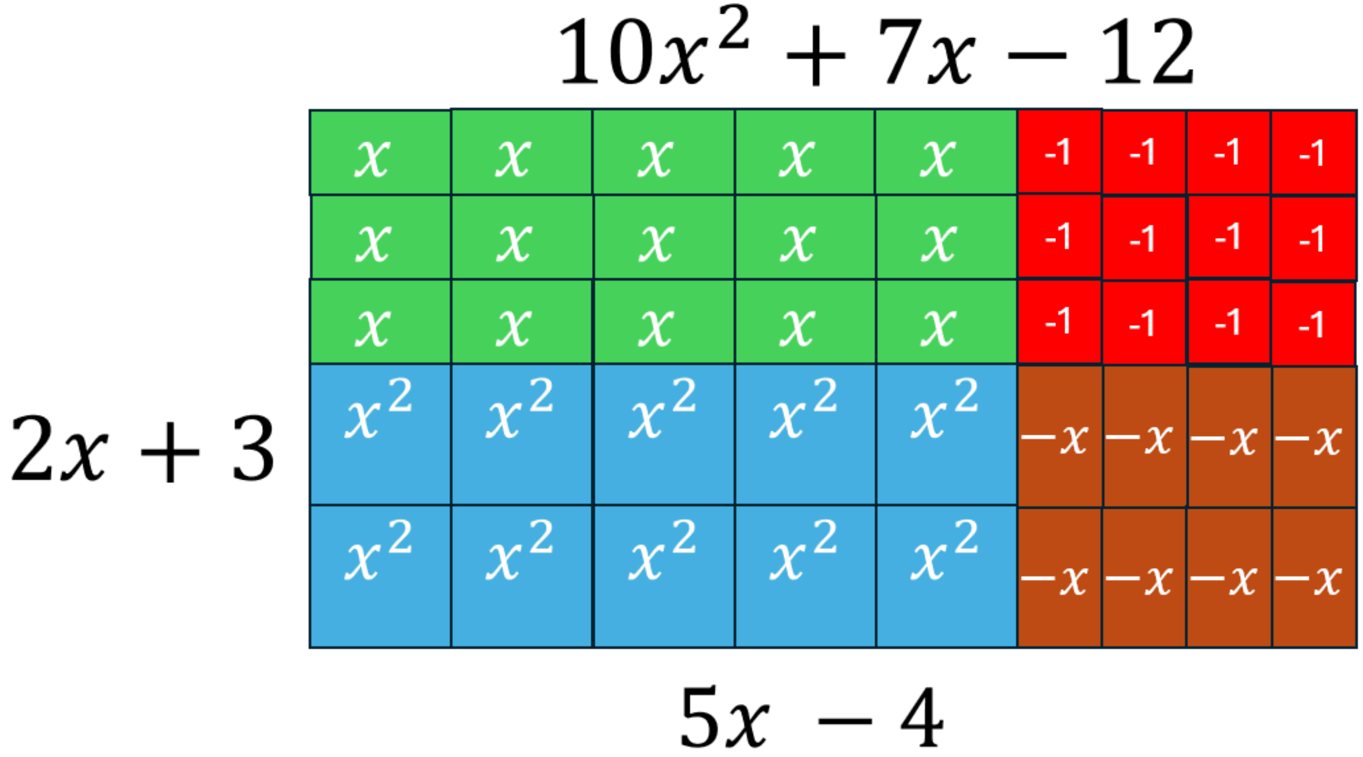
Let's factorise $15x^2 - x - 6$

- First we multiply $15 \times -6 = -90$
- Then we need to find a and b such that $a + b = -1$ and $ab = -90$
- Factor pairs of 90 (we can make either of each pair negative):
- 1, 90
- 2, 45
- 3, 30
- 5, 18
- 6, 15
- 9, 10
- Of all these, only 9 and 10 have a difference of 1
- So we write our expression as $15x^2 - 10x + 9x - 6$

So we have $15x^2 - 10x + 9x - 6$, now what?

- Well, we can factorise by grouping so let's do that first
- $15x^2 - 10x + 9x - 6 = 5x(3x - 2) + 3(3x - 2)$
- $= (5x + 3)(3x - 2)$
- This approach will also work for our example $6x^2 + 23x + 7$

Visualising



$$10x^2 + 7x - 12 = (2x + 3)(5x - 4)$$

Now I'll show you a more systematic way for when our gropuing is more confusing

- Let's take a new example: $9x^2 + 6x - 8$
- We find $9 \times -8 = -72$
- And looking at the factors, we see $12 \times (-6) = -72$ while $12 + (-6) = 6$
- So we get $9x^2 - 6x + 12x - 8$
- We've got three terms divisible by 3, how do we know which to group?

- Let's say our factorised expression is $(cx + d)(ex + f)$
- We now need c, d, e, f such that $ce = 9$, $cf = -6$, de = 12 and $df = -8$
- That sounds harder than it is
- $ce = 9$ and $cf = -6$ will have a common factor: that could be c
- Then we use that to find d, e, f
- Let's try it

$ce = 9$, $cf = -6$, de = 12 and $df = -8$

- 9 and -6 are both divisible by 3, let's say $c = 3$
- Then $e = 3$
- $de = 12$ is divisible by 3 so this is possible
- So $d = 12 \div 3 = 4$
- And $f = -8 \div 2 = -2$
- And we have our values

Now what?

- We put our values $c = 3$, $d = 4$, $e = 3$ and $f = -2$ in $(cx + d)(ex + f)$
- And get $(3x + 4)(3x - 2)$
- Let's check our answer by expanding
- By FOIL, $(3x + 4)(3x - 2) = 9x^2 + 12x - 6x - 8$
- Which is what we started with, so we know we're right
- And now we've factorised our trinomial!

Learning Intention

5F	**15. I can factorise by completing the square.** e.g. Factorise $x^2 + 6x + 2$ by completing the square.	☐
5F	**16. I can factorise non-monic quadratics by completing the square.** e.g. Factorise $2x^2 + 6x + 3$ by completing the square.	☐
5F	**17. I can recognise when a quadratic cannot be factorised.** e.g. Factorise $x^2 - 3x + 4$ by completing the square if possible.	☐

Factorising by Completing the Square

- How would I factorise $x^2 + 6x + 9$?
- Remember the formula for perfect squares?
  - $(a + b)^2 = a^2 + 2ab + b^2$
- Can I use this to factorise $x^2 + 6x + 7$?
- Let's see
- $x^2 + 6x + 7 = x^2 + 6x + 9 - 2$
  - $= (x + 3)^2 - 2$
- Now, remember the difference of squares with surds?
- $x^2 + 6x + 7 = (x + 3)^2 - 2$
  - $= (x + 3 + \sqrt(2))(x + 3 - \sqrt(2))$

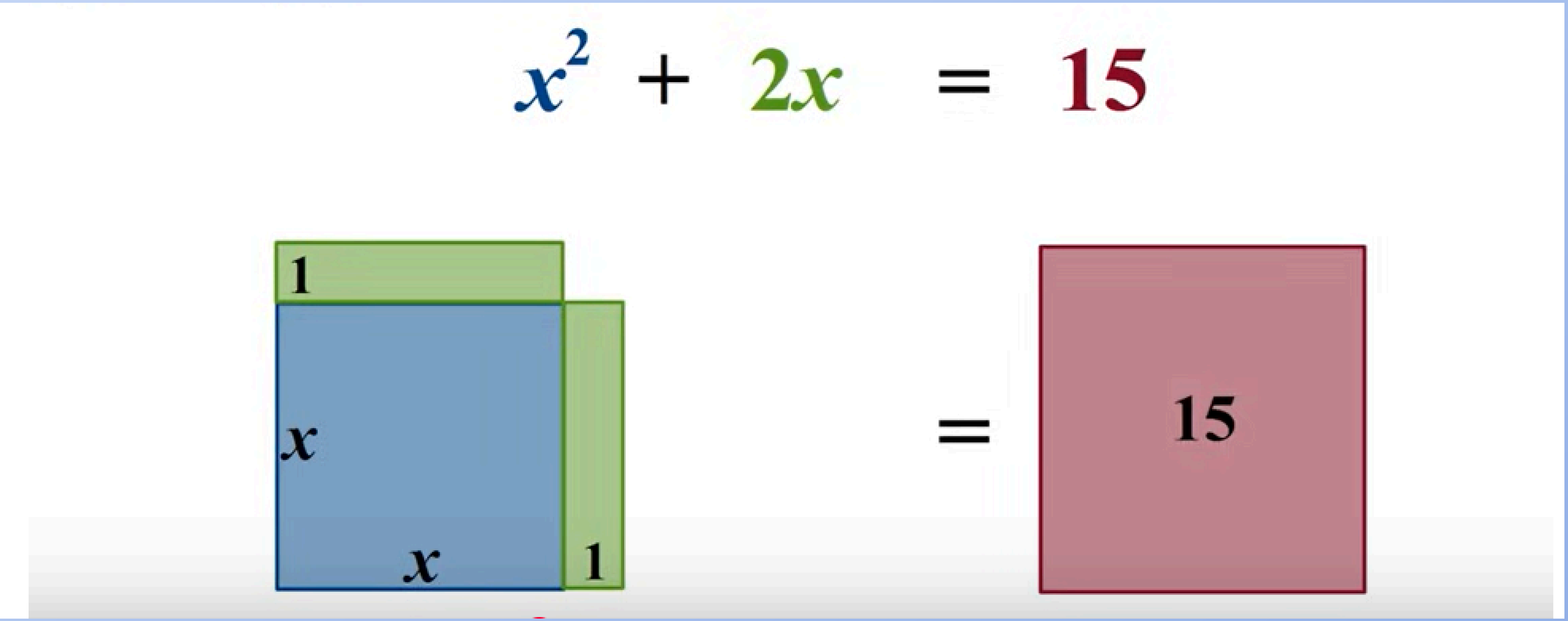
Completing the Square

- Let's say we have an expression of the form $x^2 + bx + c$
- We want the constant term to be a square
- What square though?
- We need $(\frac{b}{2})^2$ to make $(x + b)^2$
- So we add and subtract $(\frac{b}{2})^2$ (so that we don't change the expression)
- $x^2 + bx + c = x^2 + bx + (\frac{b}{2})^2 + c - (\frac{b}{2})^2$
  - $= (x + b)^2 + c - (\frac{b}{2})^2$
- And then we apply our difference of squares formula - if we can

Can we always use the difference of squares?

- We did it with $x^2 + 6x + 7$
- Let's try $x^2 + 6x + 11$
- = $(x + 3)^2 + 2$
- This isn't a difference, so we can't use our formula
- This would happen with any constant term greater than 9
- In fact, we can extend this
- When we have an expression of the form $x^2 + bx + c$
  - If $c < (\frac{b}{2})^2$, we can use the difference of squares formula
- (This is leading to something bigger later this chapter!)

Why do we complete the square?



- We want a little piece to complete the left square so it equals the right square

Learning Intention

5G	**18. I can solve a quadratic equation by factorising and applying the Null Factor Law.** e.g. Solve $3x^2 - 9x = 0$.	☐
5G	**19. I can solve a quadratic equation by first rearranging into standard form.** e.g. Solve $x^2 = 2x + 3$.	☐

Solving Quadratic Equations by Factorising

- How do we solve $(x+5)^2 = 0$?
- What about $(x+5)(x+2) = 0$?
- So, how would we solve $x^2 + 7x + 10 = 0$?

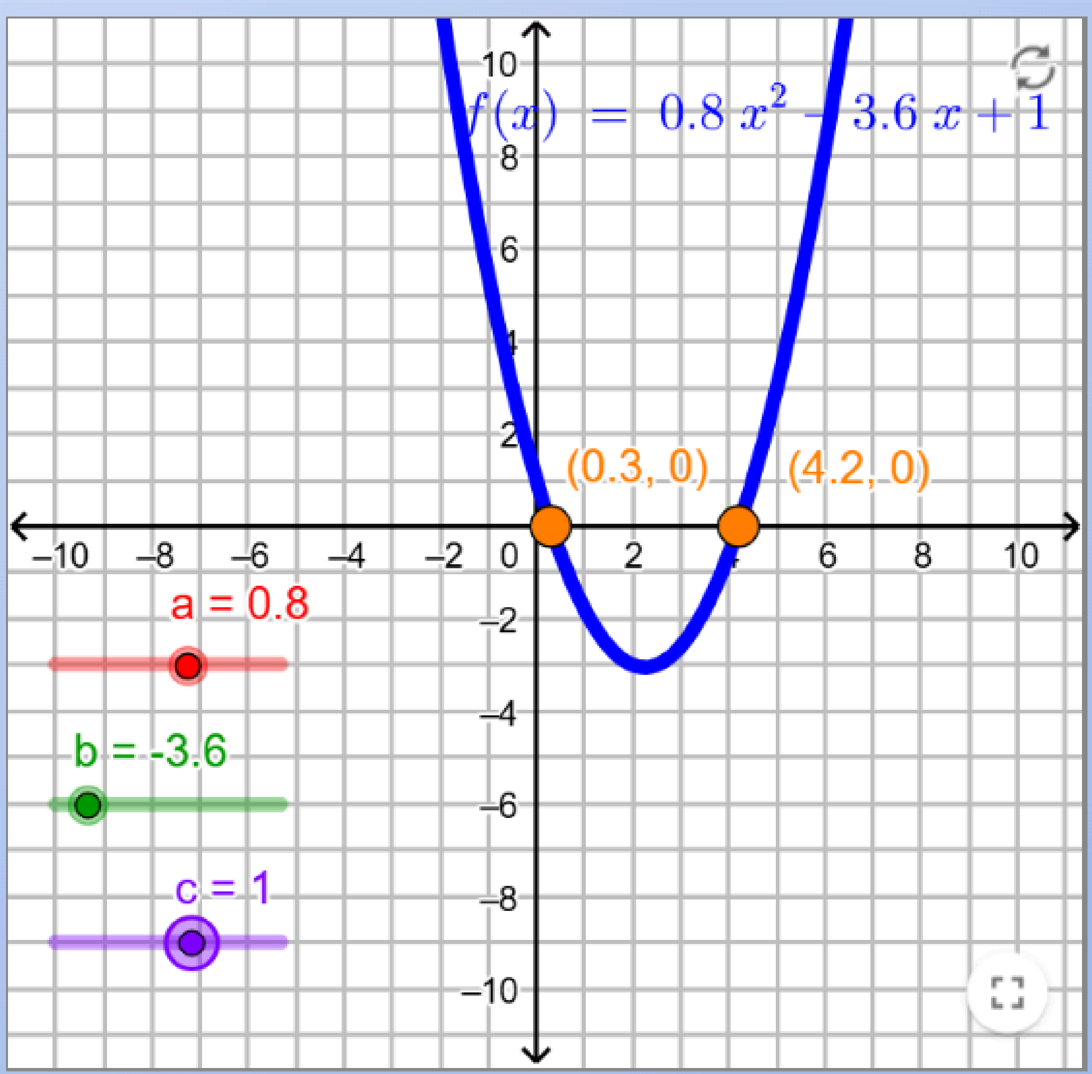
Solving Quadratic Equations by Factorising

- All quadratic equations can be written in the form $ax^2 + bx + c = 0$
- Quadratic equations have 0, 1 or 2 solutions
- Procedure:
    i. Get all the terms on the left
    ii. Factorise the left side
    iii. Setting each expression in brackets to zero will give us the solutions
        - This is by the Null Factor Law
        - Which states, if $p \times q = 0$, then either $p = 0$ or $q = 0$ or both

Example

$x^2 - x = 6$

1. We get $x^2 - x - 6 = 0$
2. We factorise: 2 - 3 = -1 and 2 × -3 = 6
   - So, $x^2 - x - 6 = (x + 2)(x - 3) = 0$
3. Now we have (x+2) = 0 or (x-3) = 0
   - So, x = -2 or x = 3



- When does the equation have solutions?
  - When the curve touches the x-axis
  - That is, when f(x) = 0

Learning Intention

5H

20. I can solve a word problem using a quadratic model.

e.g. The area of a rectangle is 60 m^2 and its length is 4 metres more than its breadth. Find the dimensions of the rectangle.

☐

Solving Problems with Quadratic Equations

I throw a paper plane into the air from a height of 3 metres: it follows a parabolic path. The equation for plane's vertical height is $s = -\frac{1}{4}t^2 + t + 3$. When does it hit the ground?

- This is an example of a quadratic equation modelling a real world problem
- The steps for using quadratic equations to solve problems are:
    - Define a variable, e.g. "Let x be the height"
    - Write an equation.
    - Solve the equation.
    - Check that the solutions seem reasonable and choose the solution accordingly, e.g. the radius of a circle can't be negative

Learning Intention

5I

21. I can solve a quadratic equation using completing the square.
e.g. Solve $x^2 + 4x - 22 = 0$ by first completing the square.

☐

Solving Equations By Completing The Square

When we can't find integer solutions, we can always complete the square

To solve an equation, we will try to factorise it first

1. Are there any common factors that can be taken out?
2. Are there any integers that multiply to give the constant term and add to give the coefficient of x?
3. If not, try completing the square
4. Now, if possible, try to factorise using the difference of squares
5. Use the Null Factor Law (if $p \times q = 0$, then either $p = 0$ or $q = 0$ or both)

Learning Intentions

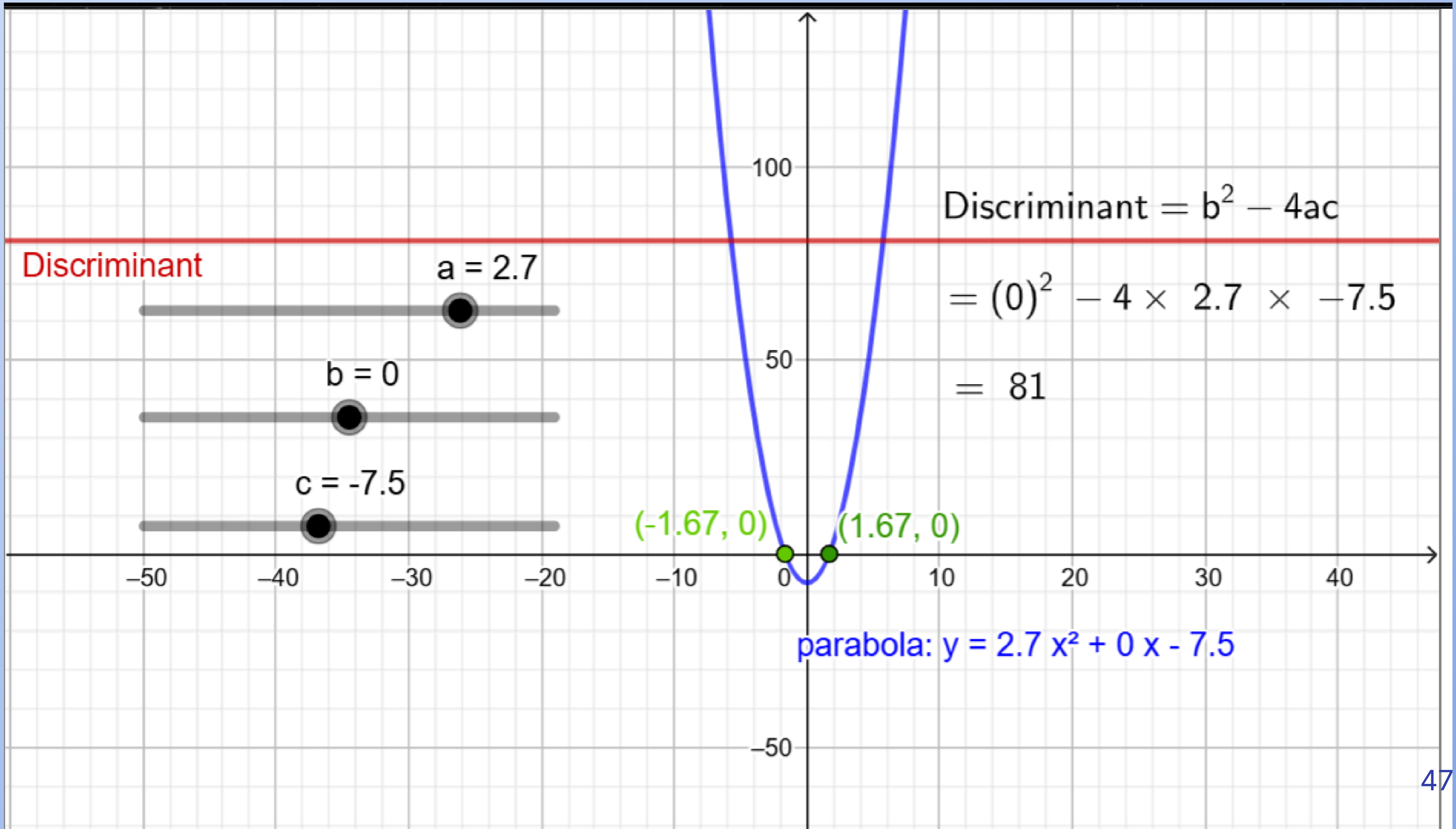
5J	**22. I can determine the number of solutions of a quadratic equation using the discriminant.** e.g. Use the discriminant to determine the number of solutions of the equation $2x^2 - 3x - 5 = 0$.	☐
5J	**23. I can use the quadratic formula to solve a quadratic equation.** e.g. Find the exact solutions of $2x^2 + 3x - 4 = 0$ using the quadratic formula.	☐

Quadratic Formula

- By now we've seen a lot of ways of solving quadratic equations
- But it can be a bit annoying having to figure out which to use
- What if we had a way that worked for all quadratic equations?
- Introducing... the Quadratic Formula
- $$\boxed{x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}}$$

How Many Solutions?

- We've said that there can be 0, 1 or 2 solutions to a quadratic equation
- But how do we know how many solutions a given equation has?
- We look at the part of the quadratic formula under the squareroot
    - The formula is $x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
    - So we look at $b^2 - 4ac$
- Remember that we can't find the squareroot of a negative number
- So if $b^2 - 4ac < 0$, there are no real solutions
- We call $b^2 - 4ac$ the **discriminant**



Discriminant = b² − 4ac
= (0)² − 4 × 2.7 × −7.5
= 81
Discriminant
a = 2.7
b = 0
c = -7.5
(-1.67, 0)
(1.67, 0)
parabola: y = 2.7 x² + 0 x - 7.5
100
50
–50
–50
–40
–30
–20
–10
0
10
20
30
40

Any quadratic equation can be written in the form:

$$ax^2 + bx + c = 0$$

Let's divide both sides by x^2's coefficient a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now let's complete the square: $\frac{b}{a}x$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 + \frac{c}{a} - (\frac{b}{2a})^2 = 0$$

Let's factorise our completed square

$$(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}) = 0$$

Let's get our constant terms with the same denominator

$$(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a^2} = 0$$

Bring our constant terms to the right so we can squareroot

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

And now let's squareroot (remember both positive and negative roots)

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$4a^2$ is a perfect square so we can bring it out

And we want x by itself so we subtract $\frac{b}{2a}$ from both sides

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Our terms have the same denominator so

$$\boxed{x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}}$$

And there we are

Chapter Summary